

Two Greedy Consequences for Maximum Induced Matchings

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Abstract

We prove that, for every integer d with $d \geq 3$, there is an approximation algorithm for the maximum induced matching problem restricted to $\{C_3, C_5\}$ -free d -regular graphs with performance ratio $0.708\bar{3}d + 0.425$, which answers a question posed by Dabrowski et al. (Theor. Comput. Sci. 478 (2013) 33-40). Furthermore, we show that every graph with m edges that is k -degenerate and of maximum degree at most d with $k < d$, has an induced matching with at least $m/((3k - 1)d - k(k + 1) + 1)$ edges.

Keywords: induced matching; greedy algorithm; approximation algorithm; strong chromatic index; degenerate graph

1 Introduction

A set M of edges of a graph G is an *induced matching* of G if the set of vertices of G that are incident with the edges in M induces a 1-regular subgraph of G , or, equivalently, if M is an independent set of the square of the line graph of G . The *induced matching number* $\nu_2(G)$ of G is the maximum cardinality of an induced matching of G . Induced matchings were introduced by Stockmeyer and Vazirani [15] as a variant of classical matchings [11]. While classical matchings are structurally and algorithmically well understood [11], it is hard to find a maximum induced matching [2, 15] and efficient algorithms are only known for special graph classes [1, 3, 5]. The problem to determine a maximum induced matching in a given graph, called MAXIMUM INDUCED MATCHING for short, is even APX-complete for bipartite d -regular graphs for every $d \geq 3$ [5, 7].

On the positive side, a natural greedy strategy applied to a d -regular graph G , which mimics the well-known greedy algorithm for the maximum independent set problem applied to the square of the line graph of G , produces an induced matching with at least $\frac{m(G)}{2d^2-2d+1}$ edges. Since every induced matching of a d -regular graph contains at most $\frac{m(G)}{2d-1}$ edges, this already yields an approximation algorithm for MAXIMUM INDUCED MATCHING in d -regular graphs with performance ratio $d - \frac{1}{2} + \frac{1}{4d-2}$ as observed by Zito [18]. This was improved slightly by Duckworth et al. [7] who describe an approximation algorithm with asymptotic performance ratio $d - 1$. The best known approximation algorithm for MAXIMUM INDUCED MATCHING restricted to d -regular graphs for general d is due to Gotthilf and Lewenstein [9] who elegantly combine a greedy strategy with a local search algorithm to obtain a performance ratio of $0.75d + 0.15$. In [10] Joos et al. describe a linear time algorithm that finds an induced matching with at least $\frac{m(G)}{9}$ edges for a given 3-regular graph G , which yields an approximation algorithm for cubic graphs with performance ratio $\frac{9}{5}$.

At the end of [5] Dabrowski et al. propose to study approximation algorithms for regular bipartite graphs, and to determine whether the above performance ratios can be improved in the bipartite case. As our main result we show that this is indeed possible.

Theorem 1 *For every integer d with $d \geq 3$, there is an approximation algorithm for MAXIMUM INDUCED MATCHING restricted to $\{C_3, C_5\}$ -free d -regular graphs with performance ratio $\frac{17}{24}d + \frac{17d}{48d-24} \leq 0.708\bar{3}d + 0.425$.*

Our proof of Theorem 1 builds on the approach of Gotthilf and Lewenstein [9], and all proofs are postponed to Section 2.

Our second result, which also relies on a greedy strategy, is a lower bound on the induced matching number of degenerate graphs. This result is related to recent bounds on the *strong chromatic index* $\chi'_s(G)$ of a graph G [8], which is defined as the minimum number of induced matchings into which the edge set of G can be partitioned. The most prominent conjecture concerning this notion was made by Erdős and Nešetřil in 1985 and states that the strong chromatic index of a graph G of maximum degree at most d is at most $\frac{5}{4}d^2$. The most significant progress towards this conjecture is due to Molloy and Reed [14] who proved $\chi'_s(G) \leq 1.998d^2$ provided that d is sufficiently large. Again a natural greedy edge coloring implies $\chi'_s(G) \leq 2d^2 - 2d + 1$.

Recall that a graph G is *k-degenerate* for some integer k , if every non-empty subgraph of G has a vertex of degree at most k . Recently, Chang and Narayanan [4] studied the strong chromatic index of 2-degenerate graphs and inspired the following results about the strong chromatic index of a

k -degenerate graph G of maximum degree at most d with $k \leq d$:

$$\chi'_s(G) \leq \begin{cases} 10d - 10 & , \text{ if } k = 2 \text{ [4]} \\ 8d - 4 & , \text{ if } k = 2 \text{ [13]} \\ (4k - 1)d - k(2k + 1) + 1 & , [6] \\ (4k - 2)d - k(2k - 1) + 1 & , [16] \\ (4k - 2)d - 2k^2 + 1 & , [17]. \end{cases} \quad (1)$$

The proofs of (1) all rely in some way on greedy colorings and (1) immediately implies that $\nu_s(G) \geq \frac{m(G)}{\chi'_s(G)} \geq \frac{m(G)}{4kd + O(k+d)}$ for the considered graphs. We show that the factor 4 can be reduced to 3.

Theorem 2 *If G is a k -degenerate graph of maximum degree at most d with $k < d$, then $\nu_s(G) \geq \frac{m(G)}{(3k-1)d - k(k+1) + 1}$.*

Before we proceed to the proofs of Theorems 1 and 2 we collect some notation and terminology.

We consider finite, simple, and undirected graphs, and use standard terminology and notation. For a graph G , we denote the vertex set, edge set, order, and size by $V(G)$, $E(G)$, $n(G)$, and $m(G)$, respectively. If G has no cycle of length 3 or 5, then G is $\{C_3, C_5\}$ -free. Let $L(G)$ denote the line graph of G and let G^2 denote the square of G . For an edge e of G , let $C_G(e) = \{e\} \cup N_{L(G)^2}(e) = \{f \in E(G) : \text{dist}_{L(G)}(e, f) \leq 2\}$ and let $c_G(e) = |C_G(e)|$. Note that a set of edges of G is an induced matching if and only if it does not contain two distinct edges e and f with $f \in C_G(e)$, or, equivalently, $e \in C_G(f)$. In a maximal induced matching M of a graph G , for every edge f in $E(G) \setminus M$, there is some edge e in M with $f \in C_G(e)$. For some edges f , the choice of e might be unique, which motivates the following definition. For a set M of edges of G , let $PC_G(M, e) = C_G(e) \setminus \bigcup_{f \in M \setminus \{e\}} C_G(f)$ and let $pc_G(M, e) = |PC_G(M, e)|$. For two disjoint sets X and Y of vertices of G , let $E_G(X, Y)$ be the set of edges uv of G with $u \in X$ and $v \in Y$, and let $m_G(X, Y) = |E_G(X, Y)|$. For a set E of edges of G , let $G - E = (V(G), E(G) \setminus E)$. For a positive integer k , let $[k] = \{1, 2, \dots, k\}$.

2 Proofs

The greedy strategy for maximum induced matching relies on the following lemma.

Lemma 3 *Let G be a graph. If G_0, \dots, G_k are such that*

- $G_0 = G$, and
- for $i \in [k]$, there is an edge e_i of G_{i-1} such that $G_i = G_{i-1} - C_{G_{i-1}}(e_i)$.

then,

- (i) *If e and f are edges of G_i for some $i \in [k]$, then $f \in C_{G_i}(e)$ if and only if $f \in C_G(e)$.*
- (ii) *The set $\{e_1, \dots, e_k\}$ is an induced matching.*

Proof: (i) Let $e = uv$ and $f = xy$ be edges of G_i for some $i \in [k]$. Since G_i is a subgraph of G , it follows that $f \in C_{G_i}(e)$ immediately implies $f \in C_G(e)$. Hence, for a contradiction, we may assume that $f \in C_G(e) \setminus C_{G_i}(e)$. Since $f \notin C_{G_i}(e)$, the two edges e and f do not share a vertex. Since $f \in C_G(e)$, we may assume that the graph G contains the edge ux . Since $f \notin C_{G_i}(e)$, the edge ux does not belong to G_i , which implies $ux \in C_{G_{j-1}}(e_j)$ for some $j \leq i$. Note that e and f are edges of

G_{j-1} . If ux is incident with e_j , then one of e and f is incident with e_j , and belongs to $C_{G_{j-1}}(e_j)$. If ux is not incident with e_j , then, by symmetry, we may assume that G_{j-1} contains an edge between u and a vertex incident with e_j , which implies that e belongs to $C_{G_{j-1}}(e_j)$. In both cases we obtain the contradiction that one of the two edges e and f does not belong to G_j and hence also not to G_i .

(ii) If $e_j \in C_G(e_i)$ for some $i, j \in [k]$ with $i < k$, then, by (i), we have $e_j \in C_{G_{i-1}}(e_i)$, which implies the contradiction that e_j does not belong to G_i and hence also not to G_{j-1} . \square

Algorithm 1, called $\text{GREEDY}(f)$, corresponds to the greedy algorithm used by Gotthilf and Lewenstein.

$\text{GREEDY}(f)$

Input: A d -regular graph G .

Output: A pair (M, G') such that M is an induced matching of G and G' is a subgraph of G .

$G_0 \leftarrow G$;

$G' \leftarrow G_0$;

$i \leftarrow 1$;

$M \leftarrow \emptyset$;

while $\min\{c_{G_{i-1}}(e) : e \in E(G_{i-1})\} \leq f$ **do**

 Choose an edge e_i of G_{i-1} with $c_{G_{i-1}}(e_i) \leq f$;

$M \leftarrow M \cup \{e_i\}$;

$G_i \leftarrow G_{i-1} - C_{G_{i-1}}(e_i)$;

$G' \leftarrow G_i$;

$i \leftarrow i + 1$;

end

return (M, G') ;

Algorithm 1: The greedy algorithm of Gotthilf and Lewenstein depending on f .

Obviously, $\text{GREEDY}(f)$ can be performed in polynomial time, and, by Lemma 3, it works correctly. The intuitive idea behind $\text{GREEDY}(f)$ is to restrict the greedy choices to edges that are not too expensive in the sense that their inclusion in the induced matching does not eliminate too many of the remaining edges. In fact, $\text{GREEDY}(f)$ adds a fraction of at least $\frac{1}{f}$ of the edges in $E(G) \setminus E(G')$ to M , which is better than the trivial fraction $\frac{1}{2d^2 - 2d + 1}$ for $f < 2d^2 - 2d + 1$. The drawback of $\text{GREEDY}(f)$ is that it might not consume all edges of G , that is, $E(G) \setminus E(G')$ might be small compared to $E(G)$.

By Lemma 3, the union of M with any induced matching of G' is an induced matching of G . Therefore Gotthilf and Lewenstein combine $\text{GREEDY}(f)$ applied to G with Algorithm 2, called LOCAL SEARCH , applied to G' . LOCAL SEARCH starts with an empty matching M' and performs the following two simple augmentation operations as long as possible:

- Add an edge from $E(G') \setminus M'$ to M' if this results in an induced matching.
- Replace one edge in M' with two edges from $E(G') \setminus M'$ if this results in an induced matching.

Clearly, LOCAL SEARCH can be performed in polynomial time, and, as observed above, the union of M with its output M' is an induced matching of G . The crucial observation is that the graph G' produced by $\text{GREEDY}(f)$ has an additional structural property. As a subgraph of G , it is trivially a graph of maximum degree at most d but additionally each of its edges e satisfies $c_{G'}(e) > f$, which allows the improved analysis of LOCAL SEARCH in the following two Lemmas 4 and 5.

LOCAL SEARCH

Input: A graph G .

Output: An induced matching M of G .

$M \leftarrow \emptyset$;

repeat

if $M \cup \{e\}$ is an induced matching of G for some edge $e \in E(G) \setminus M$ **then**

$M \leftarrow M \cup \{e\}$;

end

if $(M \setminus \{e\}) \cup \{e', e''\}$ is an induced matching of G for some three distinct edges $e \in M$ and $e', e'' \in E(G) \setminus M$ **then**

$M \leftarrow (M \setminus \{e\}) \cup \{e', e''\}$;

end

until M does not increase during one iteration;

return M ;

Algorithm 2: The local search algorithm of Gotthilf and Lewenstein.

Lemma 4 Let G be a $\{C_3, C_5\}$ -free graph of maximum degree at most d for some $d \geq 3$ such that $\min\{c_G(e) : e \in E(G)\} > \frac{17}{12}d^2$. If M is an induced matching of G produced by LOCAL SEARCH applied to G , then, for every edge $e \in M$, $pc_G(M, e) \leq \frac{5}{6}d^2 + 1$.

Proof: Since M is produced by LOCAL SEARCH, it has the following properties.

- (a) For every edge e of G , there is some edge e' in M with $e \in C_G(e')$; because otherwise $e \notin M$ and $M \cup \{e\}$ is an induced matching of G .
- (b) If e is in M and e' and e'' are two distinct edges in $PC_G(M, e)$, then $e'' \in C_G(e')$; because otherwise $e', e'' \notin M$ and $(M \setminus \{e\}) \cup \{e', e''\}$ is an induced matching of G .

Let $e = xy \in M$.

Let X be the set of neighbors u of x distinct from y such that $xu \in PC_G(M, e)$. Let Y be the set of neighbors u' of y distinct from x such that $yu' \in PC_G(M, e)$. Since G is $\{C_3, C_5\}$ -free, the sets X and Y are disjoint. Let X_2 be the set of vertices v in $V(G) \setminus (\{x, y\} \cup X \cup Y)$ such that there is some vertex u in X with $uv \in PC_G(M, e)$. Let Y_2 be the set of vertices v' in $V(G) \setminus (\{x, y\} \cup X \cup Y)$ such that there is some vertex u' in Y with $u'v' \in PC_G(M, e)$. Since G is $\{C_3, C_5\}$ -free, the sets X_2 and Y_2 are disjoint. Let X_1 be the set of vertices in X that have a neighbor in X_2 . Let Y_1 be the set of vertices in Y that have a neighbor in Y_2 . Since G is $\{C_3, C_5\}$ -free, the sets X , Y , X_2 , and Y_2 are independent, and there are no edges between X_2 and Y as well as between Y_2 and X .

By definition, $\{x, y\} \cup E_G(\{x\}, X) \cup E_G(\{y\}, Y)$ is the set of edges in $PC_G(M, e)$ that are identical or adjacent with e .

Note that if $f \in PC_G(M, e)$ is not identical or adjacent with e , then G contains an edge g that is adjacent with e and f . If $g \notin PC_G(M, e)$, then $g \in C_G(e')$ for some edge e' in M distinct from e , which implies the contradiction $f \in C_G(e')$. Hence $g \in PC_G(M, e)$. This implies that all edges in $PC_G(M, e) \setminus \{e\}$ are incident with a vertex in $X \cup Y$.

If uv is an edge such that $u \in X_1$ and $v \in X_2$, then, by definition, $xu \in PC_G(M, e)$ and $u'v \in PC_G(M, e)$ for some $u' \in X$. Clearly, $uv \in C_G(e)$. If $uv \notin PC_G(M, e)$, then $uv \in C_G(e')$ for some edge e' in M distinct from e , which implies the contradiction that one of the two edges xu and $u'v$

belongs to $C_G(e')$. Hence, by symmetry, all edges in $E_G(X_1, X_2) \cup E_G(Y_1, Y_2)$ belong to $PC_G(M, e)$. Similarly, it follows that all edges in $E_G(X, Y)$ belong to $PC_G(M, e)$, and hence

$$\begin{aligned} PC_G(M, e) &= \{x, y\} \cup E_G(\{x\}, X) \cup E_G(\{y\}, Y) \\ &\quad \cup E_G(X_1, X_2) \cup E_G(Y_1, Y_2) \cup E_G(X, Y). \end{aligned} \quad (2)$$

If $u \in X_1$ and $u' \in Y$, then there is some $v \in X_2$ such that $uv \in PC_G(M, e)$ and, by definition, $yu' \in PC_G(M, e)$. Since G is $\{C_3, C_5\}$ -free, property (b) implies that u and u' are adjacent, that is, every vertex in X_1 is adjacent to every vertex in Y , and, by symmetry, every vertex in Y_1 is adjacent to every vertex in X .

If $u_1, u_2 \in X_1$ and $v_1, v_2 \in X_2$ are four distinct vertices such that u_1v_1 and u_2v_2 are edges of G , then, by (2), $u_1v_1, u_2v_2 \in PC_G(M, e)$ and, since G is $\{C_3, C_5\}$ -free, property (b) implies that v_2 is a neighbor of u_1 or v_1 is a neighbor of u_2 , that is, the bipartite graph between X_1 and X_2 is $2K_2$ -free. This implies that the sets $N_G(u) \cap X_2$ for u in X_1 are ordered by inclusion. Hence, if X_1 is non-empty, then X_1 contains a vertex u_x that is adjacent to all vertices in X_2 . By symmetry, if Y_1 is non-empty, then the set Y_1 contains a vertex u_y that is adjacent to all vertices in Y_2 .

We consider three cases.

Case 1 X_1 and Y_1 are both non-empty.

Since u_x is adjacent to each vertex in $\{x\} \cup X_2 \cup Y$, we obtain $|X_2| + |Y| \leq d - 1$, and, by symmetry, $|Y_2| + |X| \leq d - 1$. Now (2) implies

$$\begin{aligned} pc_G(M, e) &= 1 + m_G(\{x\}, X) + m_G(\{y\}, Y) + (m_G(\{u_x\}, X_2) + m_G(X_1 \setminus \{u_x\}, X_2)) \\ &\quad + (m_G(\{u_y\}, Y_2) + m_G(Y_1 \setminus \{u_y\}, Y_2)) + m_G(X, Y) \\ &= 1 + |X| + |Y| + (|X_2| + m_G(X_1 \setminus \{u_x\}, X_2)) \\ &\quad + (|Y_2| + m_G(Y_1 \setminus \{u_y\}, Y_2)) + m_G(X, Y) \\ &\leq 2d - 1 + m_G(X_1 \setminus \{u_x\}, X_2) + m_G(Y_1 \setminus \{u_y\}, Y_2) + m_G(X, Y) \end{aligned} \quad (3)$$

Note that u_x is adjacent to all vertices in $X_2 \cup Y$ and that x is adjacent to all vertices in $\{y\} \cup (X \setminus \{u_x\})$. There are $|Y|$ edges between y and Y , $m_G(X, Y) - |Y|$ edges between $X \setminus \{u_x\}$ and Y , and $m_G(X_1 \setminus \{u_x\}, X_2)$ edges between X_2 and $X \setminus \{u_x\}$. This implies

$$\begin{aligned} c_G(xu_x) &\leq 2d^2 - 2d + 1 - m_G(N_G(x) \setminus \{u_x\}, N_G(u_x) \setminus \{x\}) \\ &\leq 2d^2 - 2d + 1 - |Y| - (m_G(X, Y) - |Y|) - m_G(X_1 \setminus \{u_x\}, X_2) \\ &= 2d^2 - 2d + 1 - (m_G(X, Y) + m_G(X_1 \setminus \{u_x\}, X_2)). \end{aligned}$$

Since, by assumption, $c_G(xu_x) > \frac{17}{12}d^2$, we obtain

$$\begin{aligned} m_G(X, Y) + m_G(X_1 \setminus \{u_x\}, X_2) &\leq \frac{7}{12}d^2 - 2d + 1, \text{ and, by symmetry,} \\ m_G(X, Y) + m_G(Y_1 \setminus \{u_y\}, Y_2) &\leq \frac{7}{12}d^2 - 2d + 1. \end{aligned} \quad (4)$$

By symmetry, we may assume that $|X_1| \geq |Y_1|$.

Since every vertex in Y_1 is adjacent to y and to all vertices in X_1 , it has at most $d - 1 - |X_1|$ neighbors in Y_2 , which implies $m_G(Y_1 \setminus \{u_y\}, Y_2) \leq (|Y_1| - 1)(d - 1 - |X_1|) \leq (|Y_1| - 1)(d - 1 - |Y_1|)$.

Note that regardless of the value of $|Y_1|$, we have $-|Y_1|^2 + d|Y_1| \leq \frac{d^2}{4}$. Together with (3) and (4) we obtain

$$\begin{aligned}
pc_G(M, e) &\leq 2d - 1 + m_G(X_1 \setminus \{u_x\}, X_2) + m_G(Y_1 \setminus \{u_y\}, Y_2) + m_G(X, Y) \\
&\leq 2d - 1 + \frac{7}{12}d^2 - 2d + 1 + (|Y_1| - 1)(d - 1 - |Y_1|) \\
&= \frac{7}{12}d^2 - |Y_1|^2 + d|Y_1| - d + 1 \\
&\leq \frac{7}{12}d^2 + \frac{1}{4}d^2 - d + 1 \\
&= \frac{5}{6}d^2 - d + 1.
\end{aligned}$$

Case 2 X_1 is non-empty but Y_1 is empty.

As in Case 1, we have $|X_2| + |Y| \leq d - 1$. Clearly, $|Y| \leq d - 1$. Now (2) implies

$$\begin{aligned}
pc_G(M, e) &= 1 + m_G(\{x\}, X) + m_G(\{y\}, Y) + m_G(\{u_x\}, X_2) + m_G(X_1 \setminus \{u_x\}, X_2) \\
&\quad + m_G(X, Y) \\
&= 1 + |X| + |Y| + |X_2| + m_G(X_1 \setminus \{u_x\}, X_2) + m_G(X, Y) \\
&\leq 2d - 1 + m_G(X_1 \setminus \{u_x\}, X_2) + m_G(X, Y).
\end{aligned}$$

Exactly as Case 1, we obtain

$$m_G(X, Y) + m_G(X_1 \setminus \{u_x\}, X_2) \leq \frac{7}{12}d^2 - 2d + 1,$$

and hence

$$\begin{aligned}
pc_G(M, e) &\leq 2d - 1 + m_G(X_1 \setminus \{u_x\}, X_2) + m_G(X, Y) \\
&\leq 2d - 1 + \frac{7}{12}d^2 - 2d + 1 \\
&\leq \frac{7}{12}d^2.
\end{aligned}$$

Since $d \geq 3$, it follows that $pc_G(M, e) \leq \frac{5}{6}d^2 - d + 1$.

Case 3 X_1 and Y_1 are both empty.

Note that in this case also both sets X_2 and Y_2 are empty. Now (2) implies

$$\begin{aligned}
pc_G(M, e) &= 1 + m_G(\{x\}, X) + m_G(\{y\}, Y) + m_G(X, Y) \\
&\leq 2d - 1 + m_G(X, Y).
\end{aligned}$$

If $m_G(X, Y) \leq \frac{5}{6}d^2 - 2d + 2$, then $pc_G(M, e) \leq \frac{5}{6}d^2 + 1$. Hence, we may assume that $m_G(X, Y) > \frac{5}{6}d^2 - 2d + 2$. Since $|X| \leq d - 1$, this implies the existence of a vertex \tilde{u}_x in X with at least

$$\frac{\frac{5}{6}d^2 - 2d + 2}{d - 1} = \frac{5}{6}d - \frac{7}{6} + \frac{5}{6(d - 1)} \geq \frac{5}{6}d - \frac{7}{6}$$

neighbors in Y . Let \tilde{Y} be the set of neighbors of \tilde{u}_x in Y . Since $|Y| \leq d-1$, we have $|Y \setminus \tilde{Y}| \leq (d-1) - (\frac{5}{6}d - \frac{7}{6}) = \frac{1}{6}d + \frac{1}{6}$ and thus

$$\begin{aligned} m_G(X \setminus \{\tilde{u}_x\}, \tilde{Y}) &= m_G(X, Y) - m_G(\{\tilde{u}_x\}, Y) - m_G(X \setminus \{\tilde{u}_x\}, Y \setminus \tilde{Y}) \\ &\geq \left(\frac{5}{6}d^2 - 2d + 2\right) - (d-1) - (d-2) \left(\frac{1}{6}d + \frac{1}{6}\right) \\ &= \frac{2}{3}d^2 - \frac{17}{6}d + \frac{10}{3}. \end{aligned}$$

Note that \tilde{u}_x is adjacent to all vertices in \tilde{Y} and that x is adjacent to all vertices in $\{y\} \cup (X \setminus \{\tilde{u}_x\})$. There are $|\tilde{Y}|$ edges between y and \tilde{Y} , and $m_G(X \setminus \{\tilde{u}_x\}, \tilde{Y})$ edges between $X \setminus \{\tilde{u}_x\}$ and \tilde{Y} . This implies

$$\begin{aligned} c_G(x\tilde{u}_x) &\leq 2d^2 - 2d + 1 - m_G(N_G(x) \setminus \{\tilde{u}_x\}, N_G(\tilde{u}_x) \setminus \{x\}) \\ &\leq 2d^2 - 2d + 1 - |\tilde{Y}| - m_G(X \setminus \{\tilde{u}_x\}, \tilde{Y}) \\ &\leq 2d^2 - 2d + 1 - \left(\frac{5}{6}d - \frac{7}{6}\right) - \left(\frac{2}{3}d^2 - \frac{17}{6}d + \frac{10}{3}\right) \\ &\leq \frac{4}{3}d^2 - \frac{7}{6}, \end{aligned}$$

contradicting the assumption $c_G(xu_x) > \frac{17}{12}d^2$.

This completes the proof. \square

Lemma 5 *Let G be a $\{C_3, C_5\}$ -free graph of maximum degree at most d for some $d \geq 3$ such that $\min\{c_G(e) : e \in E(G)\} > \frac{17}{12}d^2$. If M is an induced matching of G produced by LOCAL SEARCH applied to G , then $|M| \geq \frac{m(G)}{\frac{17}{12}d^2 - d + 1}$.*

Proof: We consider the number p of pairs (e, f) where $e \in M$ and $f \in C_G(e)$. Since $c_G(e) \leq 2d^2 - 2d + 1$, we have $p \leq (2d^2 - 2d + 1)|M|$. In order to obtain a lower bound on p , we observe that the edges f of G for which there is exactly one edge e in M with $f \in C_G(e)$ are exactly those in $\bigcup_{e \in M} PC_G(M, e)$. Hence, by Lemma 4, $p \geq 2m(G) - \sum_{e \in M} pc_G(M, e) \geq 2m(G) - (\frac{5}{6}d^2 + 1)|M|$. Both estimates for p together yield the desired bound. \square

APPROXBIP(d)

Input: A $\{C_3, C_5\}$ -free d -regular graph G .

Output: An induced matching $M \cup M'$ of G .

Apply GREEDY($\frac{17}{12}d^2$) to G and denote the output by (M, G') ;

Apply LOCAL SEARCH to G' and denote the output by M' ;

return $M \cup M'$;

Algorithm 3: An approximation algorithm for $\{C_3, C_5\}$ -free d -regular graphs.

Lemma 6 *Let G be a $\{C_3, C_5\}$ -free graph of maximum degree at most d for some $d \geq 3$. The algorithm APPROXBIP(d) (cf. Algorithm 3) applied to G produces an induced matching $M \cup M'$ of G with $|M \cup M'| \geq \frac{12m(G)}{17d^2} = \frac{m(G)}{1.416d^2}$.*

Proof: By Lemma 3, the set $M \cup M'$ produced by APPROXBIP(d) applied to G is an induced matching of G . If $m = m(G) - m(G')$, then $|M| \geq \frac{m}{\frac{17}{12}d^2}$. By Lemma 5, $|M'| \geq \frac{m(G')}{\frac{17}{12}d^2 - d + 1} \geq \frac{m(G) - m}{\frac{17}{12}d^2}$. Altogether, we obtain $|M \cup M'| \geq \frac{m(G)}{\frac{17}{12}d^2}$. \square

We are now in a position to prove Theorem 1.

Proof of Theorem 1: For the sake of completeness, we give the argument for the upper bound. If N is some induced matching of the d -regular graph G , then, for each edge e in N , the two vertices incident with e are incident with exactly $2d - 1$ edges of G . Since N is induced, these sets of $2d - 1$ edges are disjoint, which implies $|N| \leq \frac{m(G)}{2d-1}$. By Lemma 6, this implies that the performance ratio of APPROXBIP(d) applied to a $\{C_3, C_5\}$ -free d -regular graph is at most $\frac{17d^2}{12(2d-1)} = \frac{17}{24}d + \frac{17d}{48d-24}$, which completes the proof. \square

For the proof of Theorem 2, we need the following lemma.

Lemma 7 *If G is a non-empty k -degenerate graph of maximum degree at most d with $k < d$, then G has an edge e with $c_G(e) \leq (3k - 1)d - k(k + 1) + 1$.*

Proof: Let X denote the set of vertices of G of degree at most k and let $Y = V(G) \setminus X$. If two vertices u and v in X are adjacent, then $c_G(e) \leq 1 + 2(k - 1)d$. If X is an independent set, then, since the graph $G[Y]$ has a vertex v of degree at most k in $G[Y]$ but degree more than k in G , some vertex u in X is adjacent to this vertex v in Y . Each of the at most k neighbors of u has degree at most d . Since v has degree at most k in $G[Y]$, it has r neighbors in Y for some $r \leq k$, which have degree at most d . The remaining at most $d - r$ neighbors of v are all in X and hence have degree at most k . Altogether this implies that

$$\begin{aligned} c_G(uv) &\leq 1 + (k - 1)d + rd + (d - 1 - r)k \\ &= 1 + 2kd + r(d - k) - d - k \\ &\leq 1 + 2kd + k(d - k) - d - k \\ &= 1 + (3k - 1)d - k(k + 1). \end{aligned}$$

Since $2(k - 1)d \leq (3k - 1)d - k(k + 1)$, the proof is complete. \square

Proof of Theorem 2: By definition, every subgraph of G is k -degenerate and of maximum degree at most d . Therefore, by Lemma 7, the algorithm GREEDY($(3k - 1)d - k(k + 1) + 1$) applied to the graph G outputs a pair (M, G') where M is an induced matching of G and the graph G' has no edges. This implies that $|M| \geq \frac{m(G)}{(3k-1)d - k(k+1) + 1}$, which completes the proof. \square

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